

Free from spurious solutions integral equation for three-magnon bound states in 1D XXZ ferromagnet

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Abstract

A new integral equation for three-magnon bound states in XXZ spin chain is suggested. Unlike the one presented by C. K. Majumdar about 40 years ago this equation does not have spurious solutions. Such an advantage is a result of decomposition of the wave function in the Bloch basis rather than in the basis of flat waves.

1 Introduction

An integral equation for three magnon bound states in Heisenberg ferromagnet was first suggested by C. K. Majumdar [1] and then directly obtained from spin Hamiltonian by several authors [2],[3]. The main advantage of this approach is its applicability to lattices with dimensions bigger than one. The main shortcoming is a presence of spurious solutions [3],[4],[5] originated due to slightly inadequate representation for a three-magnon state

$$|3\rangle = \sum_{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3} u_{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3} \mathbf{S}_{\mathbf{n}_1}^- \mathbf{S}_{\mathbf{n}_2}^- \mathbf{S}_{\mathbf{n}_3}^- |0\rangle. \quad (1)$$

Here $|0\rangle$ is the ferromagnetic ground state ($\mathbf{S}_{\mathbf{n}_j}^+|0\rangle = 0$ and the indices \mathbf{n}_j numerate a D -dimensional lattice. Amplitudes $u_{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3}$ with $\mathbf{n}_j = \mathbf{n}_k$ for $j \neq k$ are unphysical and are separated in the Shrödinger equation from the physical ones [2],[3]. It is unphysical amplitudes that spurious solutions originate from. They have already been classified [6]. In the present paper for the case $D = 1$ we represent the three-magnon bound state in the form which contains only physical amplitudes and then obtain the corresponding integral equation.

2 Hamiltonian and Shrödinger equation

Hamiltonian of the $S = 1/2$ XXZ ferromagnetic chain

$$\hat{H} = - \sum_n \left[\frac{1}{2} \left(\mathbf{S}_n^+ \mathbf{S}_{n+1}^- + \mathbf{S}_n^- \mathbf{S}_{n+1}^+ \right) + \Delta \left(\mathbf{S}_n^z \mathbf{S}_{n+1}^z - \frac{1}{4} \right) \right]. \quad (2)$$

acts in an infinite tensor product of \mathbb{C}^2 subspaces associated with each site of the chain. Here \mathbf{S}_n^j are the $S = 1/2$ spin operators corresponding to n -th site. For a three-magnon state we suggest the following representation

$$|3\rangle = \sum_{m < n < p} e^{i(m+n+p)k/3} b_{n-m, p-n}(k) \mathbf{S}_m^- \mathbf{S}_n^- \mathbf{S}_p^- |0\rangle. \quad (3)$$

where

$$|0\rangle = \prod_n |\uparrow\rangle_n, \quad (4)$$

is the ferromagnetic ground state.

It is convenient to represent the corresponding Shrödinger equation in the form

$$\left([H_0(k) + \Delta(3 + V)] b(k) \right)_{m,n} = E b_{m,n}(k), \quad (5)$$

where

$$\begin{aligned} \left(H_0(k) b(k) \right)_{m,n} &= \frac{e^{ik/3}}{2} \left((\delta_{m,1} - 1) b_{m-1,n}(k) + (\delta_{n,1} - 1) b_{m+1,n-1}(k) - b_{m,n+1}(k) \right) \\ &+ \frac{e^{-ik/3}}{2} \left((\delta_{m,1} - 1) b_{m-1,n+1}(k) + (\delta_{n,1} - 1) b_{m,n-1}(k) - b_{m+1,n}(k) \right), \\ \left(V b(k) \right)_{m,n} &= -\delta_{m,1} b_{1,n}(k) - \delta_{n,1} b_{m,1}(k). \end{aligned} \quad (6)$$

3 Three-magnon scattering states at $\Delta = 0$

In the free ($\Delta = 0$) case Eq. (5) has the following system of k -independent Bloch solutions

$$b_{m,n}^{(0)}(\mathbf{q}) = \frac{1}{\sqrt{6}} \left(e^{i[q_1 m + q_2 n]} - e^{i[q_1 m + (q_1 - q_2)n]} + e^{i[(q_2 - q_1)m - q_1 n]} \right. \\ \left. - e^{i[(q_2 - q_1)m + q_2 n]} + e^{-i[q_2 m + (q_2 - q_1)n]} - e^{-i[q_2 m + q_1 n]} \right), \quad \mathbf{q} \equiv (q_1, q_2), \quad (7)$$

with dispersion

$$E_0(k, \mathbf{q}) = -\cos\left(\frac{k}{3} - q_1\right) - \cos\left(\frac{k}{3} + q_1 - q_2\right) - \cos\left(\frac{k}{3} + q_2\right). \quad (8)$$

It is convenient to represent the wave function (7) in a compact form

$$b_{m,n}^{(0)}(\mathbf{q}) = \frac{1}{\sqrt{6}} \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg \omega} e^{i[\omega^{(1)}(q_1)m + \omega^{(2)}(q_2)n]}, \quad (-1)^{\deg \omega_{jkl}} = \varepsilon_{jkl}. \quad (9)$$

where ω give the following representations of \mathcal{S}_3 (the permutation group of three elements)

$$\omega_{123}(q_1, q_2) = (q_1, q_2), \quad \omega_{132}(q_1, q_2) = (q_1, q_1 - q_2), \quad \omega_{213}(q_1, q_2) = (q_2 - q_1, q_2), \\ \omega_{231}(q_1, q_2) = (q_2 - q_1, -q_1), \quad \omega_{312}(q_1, q_2) = (-q_2, q_1 - q_2), \quad \omega_{321}(q_1, q_2) = (-q_2, -q_1). \quad (10)$$

From (9) follows the symmetry property

$$b_{m,n}^{(0)}(\omega(\mathbf{q})) = (-1)^{\deg \omega} b_{m,n}^{(0)}(\mathbf{q}), \quad (11)$$

which reduces Ω , the fundamental region for the parameters q_1 and q_2 . If

$$\Omega_0 : \quad 0 \leq q_1 < 2\pi, \quad 0 \leq q_2 < 2\pi, \quad (12)$$

then

$$\Omega_0 = \cup_{\omega \in \mathcal{S}_3} \omega(\Omega). \quad (13)$$

The system of functions (9) is normalized and complete

$$\sum_{m,n=1}^{\infty} \bar{b}_{m,n}^{(0)}(\mathbf{q}) b_{m,n}^{(0)}(\tilde{\mathbf{q}}) = \delta(q_1 - \tilde{q}_1) \delta(q_2 - \tilde{q}_2), \quad (14) \\ \frac{3}{2\pi^2} \int_{\Omega} \bar{b}_{\tilde{m},\tilde{n}}^{(0)}(\mathbf{q}) b_{m,n}^{(0)}(\mathbf{q}) dq_1 dq_2 = \frac{1}{(2\pi)^2} \int_0^{2\pi} dq_1 \int_0^{2\pi} dq_2 \bar{b}_{\tilde{m},\tilde{n}}^{(0)}(\mathbf{q}) b_{m,n}^{(0)}(\mathbf{q}) = \delta_{m\tilde{m}} \delta_{n\tilde{n}}. \quad (15)$$

In order to prove the normalization condition one should represent the product of wave functions in the sum (14) as follows

$$\begin{aligned}\bar{b}_{m,n}^{(0)}(\mathbf{q})b_{m,n}^{(0)}(\tilde{\mathbf{q}}) &= \frac{1}{6} \sum_{\omega, \tilde{\omega} \in \mathcal{S}_3} (-1)^{\deg \omega + \deg \tilde{\omega}} e^{-i[(\omega^{(1)}(q_1) - \tilde{\omega}^{(1)}(\tilde{q}_1))m + (\omega^{(2)}(q_2) - \tilde{\omega}^{(2)}(\tilde{q}_2))n]} \\ &= \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg \omega} g_{m,n}(\mathbf{q} - \omega(\tilde{\mathbf{q}})),\end{aligned}\quad (16)$$

where

$$\begin{aligned}g_{m,n}(\mathbf{q}) &= \frac{1}{6} \sum_{\omega \in \mathcal{S}_3} e^{-i[\omega^{(1)}(q_1)m + \omega^{(2)}(q_2)n]} = \frac{1}{6} \left(e^{-i(q_1 m + q_2 n)} + e^{-i[q_1(m+n) - q_2 n]} \right. \\ &\quad \left. + e^{-i(q_2 m - q_1(m+n))} + e^{i[q_1 m - q_2(m+n)]} + e^{i[q_2(m+n) - q_1 n]} + e^{i[q_2 m + q_1 n]} \right).\end{aligned}\quad (17)$$

Since for each $d_{m,n}$

$$\sum_{m,n>0} (d_{m,m+n} + d_{m+n,n}) = \sum_{m,n>0} d_{m,n} - \sum_{n>0} d_{n,n}, \quad (18)$$

one readily obtains from (17)

$$\begin{aligned}\sum_{m,n>0} g_{m,n}(\mathbf{q}) &= \frac{1}{6} \left[\sum_{m,n=-\infty}^{\infty} e^{i(q_1 m + q_2 n)} - \sum_{n=-\infty}^{\infty} \left(e^{i(q_1 + q_2)n} + e^{i(q_1 - q_2)n} \right) + 1 \right] \\ &= \frac{1}{6} \left[4\pi^2 \delta(q_1) \delta(q_2) - 2\pi \left(\delta(q_1 + q_2) + \delta(q_1 - q_2) \right) + 1 \right].\end{aligned}\quad (19)$$

Hence according to (16) and (19)

$$\sum_{m,n>0} \bar{b}_{m,n}^{(0)}(\mathbf{q})b_{m,n}^{(0)}(\tilde{\mathbf{q}}) = \frac{2\pi^2}{3} \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg(\omega)} \delta(q_1 - \omega^{(1)}(\tilde{q}_1)) \delta(q_2 - \omega^{(2)}(\tilde{q}_2)), \quad (20)$$

(other terms cancelled after averaging over \mathcal{S}_3). But $\omega(\tilde{\mathbf{q}}) \in \Omega$ only for $\omega = \omega_{123}$. So all terms in (20) with $\omega \neq \omega_{123}$ should be omitted and we get Eq. (14).

In order to prove completeness let us first define an action of \mathcal{S}_3 on the configuration space by the following formula

$$\omega^{(1)}(q_1)m + \omega^{(2)}(q_2)n = q_1 \omega^{(1)}(m) + q_2 \omega^{(2)}(n). \quad (21)$$

According to (10) and (21)

$$\begin{aligned}\omega_{123}(m, n) &= (m, n), \quad \omega_{132}(m, n) = (m - n, -n), \quad \omega_{213}(m, n) = (-m, m + n), \\ \omega_{231}(m, n) &= (-m - n, m), \quad \omega_{312}(m, n) = (n, -m - n), \quad \omega_{321}(m, n) = (-n, -m).\end{aligned}\quad (22)$$

In this notations

$$\bar{b}_{\tilde{m},\tilde{n}}^{(0)}(\mathbf{q})b_{m,n}^{(0)}(\mathbf{q}) = \frac{1}{6} \sum_{\omega, \tilde{\omega} \in S_3} (-1)^{\deg \omega + \deg \tilde{\omega}} e^{i[q_1(\omega^{(1)}(m) - \tilde{\omega}^{(1)}(\tilde{m})) + q_2(\omega^{(2)}(n) - \tilde{\omega}^{(2)}(\tilde{n}))]} \quad (23)$$

So

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} dq_1 \int_0^{2\pi} dq_2 \bar{b}_{\tilde{m},\tilde{n}}^{(0)}(\mathbf{q})b_{m,n}^{(0)}(\mathbf{q}) = \sum_{\omega \in S_3} (-1)^{\deg \omega} \delta_{m,\omega^{(1)}(\tilde{m})} \delta_{n,\omega^{(2)}(\tilde{n})}. \quad (24)$$

Since $m, n, \tilde{m}, \tilde{n} > 0$ Eq. (15) follows from (22) and (24).

4 Shrödinger equation in the \mathbf{q} -space

Let

$$b_{m,n}(k) = \frac{3}{2\pi^2} \int_{\Omega} b_{m,n}^{(0)}(\mathbf{q})b(k, \mathbf{q})d\mathbf{q}, \quad (25)$$

or equivalently

$$b(k, \mathbf{q}) = \sum_{m,n>0} \bar{b}_{m,n}^{(0)}(\mathbf{q})b_{m,n}(k), \quad (26)$$

be a decomposition of the wave function in the basis of Bloch functions. In this framework Eq. (5) takes the form

$$(E_0(k, \mathbf{q}) + 3\Delta - E)b(k, \mathbf{q}) = -6\Delta \int_{\Omega} V(\mathbf{q}, \tilde{\mathbf{q}})b(k, \tilde{\mathbf{q}})d\tilde{\mathbf{q}}, \quad (27)$$

where

$$V(\mathbf{q}, \tilde{\mathbf{q}}) = -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \left(\bar{b}_{1,n}^{(0)}(\mathbf{q})b_{1,n}(\tilde{\mathbf{q}}) + \bar{b}_{n,1}^{(0)}(\mathbf{q})b_{n,1}(\tilde{\mathbf{q}}) \right). \quad (28)$$

As it follows from (11), (25) and (28)

$$b(k, \omega(\mathbf{q})) = (-1)^{\deg \omega} b(k, \mathbf{q}), \quad V(\omega(\mathbf{q}), \tilde{\mathbf{q}}) = V(\mathbf{q}, \omega(\tilde{\mathbf{q}})) = (-1)^{\deg \omega} V(\mathbf{q}, \tilde{\mathbf{q}}). \quad (29)$$

Hence the integration over Ω in Eq. (27) may be extended to integration over Ω_0

$$(E_0(k, \mathbf{q}) + 3\Delta - E(k))b(k, \mathbf{q}) = -\Delta \int_0^{2\pi} d\tilde{q}_1 \int_0^{2\pi} d\tilde{q}_2 V(\mathbf{q}, \tilde{\mathbf{q}})b(k, \tilde{\mathbf{q}}), \quad \mathbf{q} \in \Omega_0. \quad (30)$$

For scattering states

$$E_{scatt}(k, \mathbf{q}) = E_0(k, \mathbf{q}) + 3\Delta, \quad (31)$$

and hence the right side of (30) should be zero. For bound states we assume

$$E_{\text{bound}}(k) < \min_{\mathbf{q}}(E_{\text{scatt}}(k, \mathbf{q})) = 3\Delta - 3\cos\frac{k}{3}. \quad (32)$$

Let us now calculate $V(\mathbf{q}, \tilde{\mathbf{q}})$. According to (16) and (28)

$$V(\mathbf{q}, \tilde{\mathbf{q}}) = -\frac{1}{4\pi^2} \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg \omega} \sum_{n=1}^{\infty} g_{1,n}(\omega(\mathbf{q}) - \tilde{\mathbf{q}}) + g_{n,1}(\omega(\mathbf{q}) - \tilde{\mathbf{q}}). \quad (33)$$

As it follows from (17)

$$\begin{aligned} \sum_{n=1}^{\infty} g_{1,n}(\mathbf{q}) + g_{n,1}(\mathbf{q}) &= \frac{1}{6} \sum_{n=1}^{\infty} \left[e^{-iq_1} \left(e^{-iq_2 n} + e^{iq_2(n+1)} + e^{i(q_2-q_1)n} + e^{i(q_1-q_2)(n+1)} \right) \right. \\ &\quad \left. + e^{i(q_1-q_2)} \left(e^{iq_1 n} + e^{-iq_1(n+1)} + e^{-iq_2 n} + e^{iq_2(n+1)} \right) + e^{iq_2} \left(e^{i(q_2-q_1)n} + e^{i(q_1-q_2)(n+1)} \right) \right. \\ &\quad \left. + e^{iq_1 n} + e^{-iq_1(n+1)} \right] = \frac{\pi}{3} \sum_{\omega \in \mathcal{S}_3} e^{i\omega^{(1)}(q_1)} \delta(\omega^{(2)}(q_2)) - \frac{2}{3} \left(\cos q_1 + \cos q_2 + \cos(q_1 - q_2) \right). \end{aligned} \quad (34)$$

Eqs. (33) and (34) result in

$$V(\mathbf{q}, \tilde{\mathbf{q}}) = -\frac{1}{12\pi} \sum_{\omega, \tilde{\omega} \in \mathcal{S}_3} (-1)^{\deg \omega - \deg \tilde{\omega}} e^{i[\omega^{(1)}(q_1) - \tilde{\omega}^{(1)}(\tilde{q}_1)]} \delta[\omega^{(2)}(q_2) - \tilde{\omega}^{(2)}(\tilde{q}_2)]. \quad (35)$$

A substitution of (35) into (30) and evaluation of the sum over $\tilde{\omega}$ using appropriate changes of $\tilde{\mathbf{q}}$ gives for a bound state

$$\begin{aligned} (E_0(k, \mathbf{q}) + 3\Delta - E_{\text{bound}}(k))b(k, \mathbf{q}) &= \frac{\Delta}{2\pi} \int_0^{2\pi} d\tilde{q}_1 \int_0^{2\pi} d\tilde{q}_2 \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg \omega} \\ &\quad \cdot e^{i[\omega^{(1)}(q_1) - \tilde{q}_1]} \delta[\omega^{(2)}(q_2) - \tilde{q}_2] b(k, \tilde{\mathbf{q}}), \end{aligned} \quad (36)$$

or equivalently

$$(E_0(k, \mathbf{q}) + 3\Delta - E_{\text{bound}}(k))b(k, \mathbf{q}) = \Delta \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg \omega} e^{i\omega^{(1)}(q_1)} \psi(k, \omega^{(2)}(q_2)), \quad (37)$$

where

$$\psi(k, q) = \frac{1}{2\pi} \int_0^{2\pi} d\tilde{q} e^{-i\tilde{q}} b(k, \tilde{q}, q). \quad (38)$$

Eqs. (37) and (38) result in an integral equation

$$\psi(k, q) = \frac{\Delta}{2\pi} \sum_{\omega \in \mathcal{S}_3} (-1)^{\deg \omega} \int_0^{2\pi} \frac{e^{i(\omega^{(1)}(\tilde{q}) - \tilde{q})}}{E_0(k, \tilde{q}, q) + 3\Delta - E_{\text{bound}}(k)} \psi(k, \omega^{(2)}(q)) d\tilde{q}, \quad (39)$$

or in an extended form

$$\begin{aligned}\psi(k, q) &= \frac{\Delta}{2\pi} \int_0^{2\pi} \frac{d\tilde{q}}{E_0(k, \tilde{q}, q) + 3\Delta - E_{\text{bound}}(k)} \left[\left(1 - e^{i(q-2\tilde{q})}\right) \psi(k, q) \right. \\ &\quad \left. + \left(e^{i(q-2\tilde{q})} - e^{-i(q+\tilde{q})}\right) \psi(k, -\tilde{q}) + \left(e^{-i(q+\tilde{q})} - 1\right) \psi(k, \tilde{q} - q) \right].\end{aligned}\quad (40)$$

For future simplification we notice that

$$E_0(k, \tilde{q}, q) + 3\Delta - E_{\text{bound}}(k) = \frac{\gamma(k, q)(\tilde{z} - z_+)(\tilde{z} - z_-)}{\tilde{z}(z_- - z_+)}, \quad (41)$$

where

$$\tilde{z} = e^{i\tilde{q}}, \quad z_{\pm} = \frac{3\Delta - E_{\text{bound}}(k) - \cos(k/3 + q) \pm \gamma(k, q)}{(e^{i(k/3-q)} + e^{-ik/3})}. \quad (42)$$

and

$$\gamma(k, q) = \sqrt{\left(\cos\left(\frac{k}{3} + q\right) + E_{\text{bound}}(k) - 3\Delta\right)^2 - \left|e^{i(k/3-q)} + e^{-ik/3}\right|^2}. \quad (43)$$

From (42), (43) and (32) follows that

$$z_+ \bar{z}_- = \bar{z}_+ z_- = 1, \quad |z_+| > 1, \quad |z_-| < 1. \quad (44)$$

Hence from (41) and (44) one readily gets

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - e^{i(q-2\tilde{q})})d\tilde{q}}{E_0(k, \tilde{q}, q) + 3\Delta - E_{\text{bound}}(k)} = \frac{1 - e^{iq} \bar{z}_-^2}{\gamma(k, q)}. \quad (45)$$

Now using (45) and the following formulas

$$E_0(k, -\tilde{q}, q) = E_0(k, q + \tilde{q}, q) = E_0(k, -q, \tilde{q}), \quad (46)$$

one may reduce (40) to the form

$$\left(1 - \frac{\Delta(1 - e^{iq} \bar{z}_-^2)}{\gamma(k, q)}\right) \psi(k, q) = \frac{\Delta}{2\pi} \int_0^{2\pi} \frac{[e^{i(q+2\tilde{q})} - e^{i(\tilde{q}-q)} + e^{-i(\tilde{q}+2q)} - 1] \psi(k, \tilde{q}) d\tilde{q}}{E_0(k, -q, \tilde{q}) + 3\Delta - E_{\text{bound}}(k)}, \quad (47)$$

or

$$\left(1 - \frac{\Delta(1 - e^{iq} \bar{z}_-^2)}{\gamma(k, q)}\right) \varphi(k, q) = \frac{\Delta}{\pi} \int_0^{2\pi} \frac{[\cos 3(q + \tilde{q})/2 - \cos(q - \tilde{q})/2] \varphi(k, \tilde{q}) d\tilde{q}}{E_0(k, -q, \tilde{q}) + 3\Delta - E_{\text{bound}}(k)}, \quad (48)$$

where

$$\varphi(k, q) = e^{iq/2} \psi(k, q). \quad (49)$$

Eq. (48) is the main result of the paper. Although it is analogous to Eq. (91) of Ref. 1, its derivation does not need introduction of any additional constructions (as it has been for the Majumdar equation [2],[3]). In the Bloch basis Eq. (48) directly follows from representation (3). According to (26), (38) and (49) up to a normalization constant

$$\varphi(k, q) = \sum_{n=1}^{\infty} b_{n,1}(k) e^{iq(n+1/2)} + b_{1,n}(k) e^{-iq(n+1/2)}. \quad (50)$$

5 Checking on the Bethe Ansatz result

An exact form of the three-magnon wave function is well known [7]. Namely

$$b_{m,n}(k) = z_1^{m-1}(k)z_2^{n-1}(k), \quad (51)$$

where

$$z_1(k) = \frac{1}{4\Delta^2 - 1} \left(2\Delta e^{ik/3} + e^{-2ik/3} \right), \quad z_2(k) = \bar{z}_1(k) = \frac{1}{4\Delta^2 - 1} \left(2\Delta e^{-ik/3} + e^{2ik/3} \right). \quad (52)$$

In fact it may be readily proved that (51) satisfies the system (6) with

$$\begin{aligned} E_{bound}(k) &= 3\Delta - \frac{e^{-ik/3}}{2} \left(z_1(k) + \frac{1}{z_2(k)} + \frac{z_2(k)}{z_1(k)} \right) - \frac{e^{ik/3}}{2} \left(z_2(k) + \frac{1}{z_1(k)} + \frac{z_1(k)}{z_2(k)} \right) \\ &= 3\Delta - \frac{8\Delta^3 + \cos k}{4\Delta^2 - 1}. \end{aligned} \quad (53)$$

According to (50) and (51)

$$\varphi(k, q) = \frac{e^{iq/2}}{e^{-iq} - z_1(k)} + \frac{e^{-iq/2}}{e^{iq} - z_2(k)}. \quad (54)$$

Since

$$E_0(k, -q, \tilde{q}) + 3\Delta - E_{bound}(k) = \frac{\gamma(k, q)(\tilde{z} - \bar{z}_+)(\tilde{z} - \bar{z}_-)}{\tilde{z}(\bar{z}_- - \bar{z}_+)}, \quad (55)$$

Eq. (48) takes the form

$$\begin{aligned} \left(\gamma(k, q) + \Delta(e^{iq}\bar{z}_-^2 - 1) \right) \left(\frac{e^{iq/2}}{e^{-iq} - z_1(k)} + \frac{e^{-iq/2}}{e^{iq} - z_2(k)} \right) &= \frac{\Delta(\bar{z}_- - \bar{z}_+)}{2\pi i} \oint \left(e^{3iq/2}\tilde{z}^2 \right. \\ &\quad \left. - e^{-iq/2}\tilde{z} - e^{iq/2} + e^{-3iq/2}\frac{1}{\tilde{z}} \right) \left(\frac{\tilde{z}}{1 - \tilde{z}z_1(k)} + \frac{1}{\tilde{z}(\tilde{z} - z_2(k))} \right) \frac{d\tilde{z}}{(\tilde{z} - \bar{z}_+)(\tilde{z} - \bar{z}_-)}. \end{aligned} \quad (56)$$

Integral in the right side of (56) may be decomposed and readily calculated

$$\begin{aligned} \frac{\bar{z}_- - \bar{z}_+}{2\pi i} \oint \left(e^{3iq/2}\tilde{z}^3 - e^{-iq/2}\tilde{z}^2 - e^{iq/2}\tilde{z} + e^{-3iq/2} \right) \frac{d\tilde{z}}{(1 - \tilde{z}z_1(k))(\tilde{z} - \bar{z}_+)(\tilde{z} - \bar{z}_-)} \\ = \left(e^{3iq/2}\bar{z}_-^3 - e^{-iq/2}\bar{z}_-^2 - e^{iq/2}\bar{z}_- + e^{-3iq/2} \right) \frac{1}{1 - \bar{z}_-z_1(k)}, \end{aligned} \quad (57)$$

and ($w = 1/\tilde{z}$)

$$\begin{aligned} \frac{\bar{z}_- - \bar{z}_+}{2\pi i} \oint \left(e^{3iq/2}\tilde{z} - e^{-iq/2} - e^{iq/2}\frac{1}{\tilde{z}} + e^{-3iq/2}\frac{1}{\tilde{z}^2} \right) \frac{d\tilde{z}}{(\tilde{z} - z_2(k))(\tilde{z} - \bar{z}_+)(\tilde{z} - \bar{z}_-)} \\ = \frac{\bar{z}_- - \bar{z}_+}{2\pi i} \oint \left(e^{3iq/2} - e^{-iq/2}w - e^{iq/2}w^2 + e^{-3iq/2}w^3 \right) \frac{dw}{(1 - wz_2(k))(1 - \bar{z}_+w)(1 - \bar{z}_-w)} \\ = \left(e^{3iq/2} - e^{-iq/2}\frac{1}{\bar{z}_+} - e^{iq/2}\frac{1}{\bar{z}_+^2} + e^{-3iq/2}\frac{1}{\bar{z}_+^3} \right) \frac{\bar{z}_+}{\bar{z}_+ - z_2(k)}, \end{aligned} \quad (58)$$

Using (57), (58) and (44) one may reduce Eq. (56) to the form

$$\left(\frac{\gamma(k, q)}{\Delta} + e^{iq} \bar{z}_-^2 - 1 \right) \left(\frac{e^{iq/2}}{e^{-iq} - z_1(k)} + \frac{e^{-iq/2}}{e^{iq} - z_2(k)} \right) = \frac{e^{-3iq/2} (e^{iq} \bar{z}_-^2 - 1) (e^{2iq} \bar{z}_- - 1)}{1 - \bar{z}_- z_1(k)} + \frac{e^{3iq/2} (e^{-iq} z_-^2 - 1) (e^{-2iq} z_- - 1)}{1 - z_- z_2(k)}, \quad (59)$$

which may be checked directly by rather cumbersome calculation.

6 Conclusions

In the present paper we obtained an integral equation (48) for three-magnon bound states in 1D Heisenberg ferromagnet. The suggested equation is based on the representation (3) which does not contain unphysical amplitudes related to spurious solutions. The derivation is based on the decomposition (25) of the wave function in the basis of Bloch wave functions. The obtained equation was directly tested on the Bethe Ansatz solution (51). Basing on this result we suggest that for a study of bound states in a complex model it is better to decompose a wave function not in the flat waves basis but in a basis of states related to an appropriate solvable model.

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